Alfven modes driven nonlinearly by metric perturbations in Anisotropic Magnetized Cosmologies

Apostolos Kuiroukidis[1,2], Kostas Kleidis[1,3] and Demetrios B. Papadopoulos[1]
[1] *Department of Physics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece*
[2] *Department of Informatics, Technological Education Institute of Serres, 62124 Serres, Greece*
[3] *Department of Civil Engineering, Technological Education Institute of Serres, 62124 Serres, Greece*
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We consider anisotropic magnetized cosmologies filled with conductive plasma fluid and study the implications of metric perturbations that propagate parallel to the ambient magnetic field. It is known that in the first order (linear) approximation with respect to the amplitude of the perturbations no electric field and density perturbations arise. However when we consider the non-linear coupling of the metric perturbations with their temporal derivatives, certain classes of solutions can induce steeply increasing in time, electric field perturbations. This is verified both numerically and analytically. The source of these perturbations can be either high-frequency quantum vacuum fluctuations, driven by the cosmological pump field, in the early stages of the evolution of the Universe, or astrophysical processes, or a non-linear isotropization process, of an initially anisotropic cosmological spacetime.

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I. INTRODUCTION

Magnetic fields are known to have a widespread presence in our Universe, being a common property of the intergalactic medium in galaxy clusters [1], while, reports on Faraday rotation imply significant magnetic fields in condensations at high redshifts [2]. Studies of large-scale magnetic fields and their potential implications for the formation and the evolution of the observed structures, have been the subject of continuous investigation (see e.g. [3]-[12] for a representative though incomplete list). Magnetic fields observed in galaxies and galaxy clusters are in energy equipartition with the gas and the cosmic rays [13]. The origin of these fields, whether of astrophysical or cosmological origin, remains an unresolved issue.

If magnetism has a cosmological origin, as observations of μG fields in galaxy clusters and high-redshift protogalaxies seem to suggest, it could have affected the evolution of the Universe [14]. There are several scenarios for the generation of primordial magnetic fields (see e.g. [15]). Most of the early treatments were Newtonian, while relativistic treatments appeared recently in the literature. A common factor in all these approaches is the MHD approximation, namely the assumption that the magnetic field lines are effectively frozen in an infinitely conducting cosmic medium (i.e., of zero resistivity). With few exceptions (e.g. [16],[17],[18]) the role of non-zero resistivity and kinetic viscosity have been ignored, these features however being essential for a comprehensive picture of the non-linear magnetic field evolution. The electric fields associated with the resistivity can be the source for particle acceleration, while the induced non-linear currents may react back upon the magnetic field [18].

Many recent studies use a Newtonian or a FRW cosmological model to represent the Universe and super-impose a large-scale ordered magnetic field. The magnetic field is assumed to be too weak to destroy the FRW isotropy and the anisotropy, induced by it, is treated as a perturbation [5], [11], [19]. Current observations provide a strong motivation for the adoption of a FRW model, but the uncertainities of the cosmological *Standard Model* are several. Therefore the approximation of neglecting the background anisotropy and magnetic fields, may lead to effects and phenomena that are absent in the above treatments. Within this context the formation of small-scale structures and the excitation of resistive instabilities in Bianchi-type models has been explored long ago [16], but the issue of excitation of MHD modes in anisotropic cosmological models and their subsequent temporal evolution is far from being exhausted [20].

Usually one assumes that h^2 is small. However one may consider strong GWs either in the Early Universe when dynamical isotropization of an anisotopic cosmological model takes place, or due to amplification of quantum vacuum fluctuations from the the variable gravitational field (cosmological pump field) of the Universe (metric tensor). Also $h\dot{h}$ can be large due to high frequency gravitational wave perturbations in the Early Universe.

On the other hand examples of electrovacuum cylindically symmetric spacetimes, interacting with GWs, in the full theory are well known ([22], [23]), or the extraction of GWs by black hole collisions [24] whereas the covariant decomposition into scalar, vector and tensor perturbations is treated in [25]. Gravitational waves can carry a large amount of energy near the sources where they are generated (see e.g. [26]). Though they do not interact much with matter under normal conditions, in the linear level, it has been shown that they can excite various kinds of plasma waves, more efficiently with increasing background magnetic field (see e.g. [21], [27], [28], [29], [30], [31], [32], [33]). Non-linear effects have been studied in [27], [33].

However the problem of the behaviour of conductive

plasmas, in anisotropic magnetised cosmologies, which are driven by external perturbations, in the full non-linear level is more or less open. Motivated by this fact we consider this problem in the background of Thorne's class of anisotropic magnetized cosmologies. We write the perturbed Einstein's equations in a manner that takes into account the non-linear coupling of the metric perturbations with their temporal derivatives. Then we study mainly the electric field perturbations induced by classes of solutions, to the evolution equations of the metric perturbations (which however are written in the linear approximation). The organization of this paper is as follows: In Section II we present the derivation of the closed set of equations used, aided by the two appendices. In Section III we present the various classes of analytical solutions that enter in our discussion, classified into small and large-t solutions. In Section IV we present the numerical results of our paper and finally in Section V we present a brief discussion of the results.

II. BASIC EQUATIONS

We consider classes of anisotropic magnetized cosmologies with mater content in the form of a perfect fluid and with the magnetic field in the z-direction. The metric is taken to be

$$(g_{TT})_{\mu\nu} = \begin{pmatrix} -1, & 0, & 0, & 0 \\ 0, & A^2+h_+, & h_\times, & 0 \\ 0, & h_\times, & A^2-h_+, & 0 \\ 0, & 0, & 0, & W^2 \end{pmatrix} \qquad (1)$$

where $A = A(t)$, $W = W(t)$, for the metric components of the background spacetime (see Appendix A for the class of Thorne's anisotropic magnetized cosmologies that is used), while the metric perturbations are assumed of the form $h_+ = h_+(t,z)$ and $h_\times = h_\times(t,z)$.

The energy-momentum tensor for the perfect fluid is taken as $T^{(fl)}_{\mu\nu} = T^{(0)}_{\mu\nu} + (\delta T_{\mu\nu})$ where of course in this expansion we use the background value
$T^{(0)}_{\mu\nu} = diag(\rho(t), A^2p(t), A^2p(t), W^2(t)p(t))$ and the perturbation is computed as

$$(\delta T_{\mu\nu}) = \begin{pmatrix} \delta\rho, & \delta T_{0x}, & \delta T_{0y}, & \delta T_{0z} \\ \delta T_{0x}, & A^2\delta p + ph_+, & ph_\times, & 0 \\ \delta T_{0y}, & ph_\times, & A^2\delta p - ph_+, & 0 \\ \delta T_{0z}, & 0, & 0, & W^2\delta p \end{pmatrix} \qquad (2)$$

Here we have $\delta\rho = \delta\rho(t,z)$, $\delta p = \delta p(t,z)$, for the diagonal terms, $\delta T_{0x} = -(\rho+p)A^2(\delta u^x)$, $\delta T_{0y} = -(\rho+p)A^2(\delta u^y)$ and $\delta T_{0z} = -(\rho+p)W^2(\delta u^z)$. Also we have used the four-velocity perturbations around the background (comoving) value $u^\mu_{(0)} = (1,0,0,0)$ and perturbation of the condition $u^\mu u_\mu = -1$ ensures $\delta u^0 = 0$.

The energy-momntum tensor for the EM-field is constructed from the field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0, & E^x, & E^y, & E^z \\ -E^x, & 0, & B^z, & -B^y \\ -E^y, & -B^z, & 0, & B^x \\ -E^z, & B^y, & -B^x, & 0 \end{pmatrix} \qquad (3)$$

where $E^j = F^{j\mu}u_\mu$, $B^k = (1/2)\epsilon^{kbcd}u_bF_{cd}$ are the electric and magnetic field respectively, and the background magnetic field is assumed (as in Appendix A) to be $\vec{B}_0(t) = B_0(t)\hat{z}$. In general we assume the perturbations $\delta E^j = \delta E^j(t,z)$ and $\delta B^j = \delta B^j(t,z)$, $(j = x,y,z)$ and the energy-momentum tensor is

$$\begin{aligned} T^{\alpha\beta}_{(em)} &= \frac{1}{4\pi}[F^{\alpha\mu}F^{\beta\nu}g_{\mu\nu} - \frac{1}{4}g^{\alpha\beta}F^2], \\ (F^2 &:= F^{\alpha\mu}F^{\beta\nu}g_{\alpha\beta}g_{\mu\nu}) \end{aligned} \qquad (4)$$

For the metric of Eq. (1) the components of the Einstein tensor are computed. Then these are expanded up to the second order and we set $(G = 1 = c)$

$$\begin{aligned} G_{\mu\nu} &:= G^{(0)}_{\mu\nu} + \delta G_{\mu\nu} + \mathcal{O}(h^3) = \\ &= 8\pi(T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}) \end{aligned} \qquad (5)$$

Here on the r.h.s we have the total energy-momentum tensor, both form the fluid and the EM field, expanded with respect to the fluid and EM-field perturbations. The $\delta G_{\mu\nu}-$term contains in general first and second order corrections, with resect ot the GW amplitude, (though here the first order terms vanish as it is well known for the Alfven modes [21]) and we set, as indicated below,
$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}$. Also equating the zeroth order terms we obtain the usual Einstein's equation for the background

$$\begin{aligned} \left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{\dot{A}\dot{W}}{AW}\right) &= 8\pi\rho + A^4(B_0)^2 \\ -\frac{\ddot{A}}{A} - \frac{\ddot{W}}{W} - \left(\frac{\dot{A}\dot{W}}{AW}\right) &= 8\pi p + A^4(B_0)^2 \\ -2\frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2 &= 8\pi p - A^4(B_0)^2 \end{aligned} \qquad (6)$$

It is easy to show that the class of cosmological models of Appenix A, satisfies identically Eqs. (6). Now since we have $\nabla^\nu(G^{(0)}_{\mu\nu} + \delta G_{\mu\nu}) = 0 + \mathcal{O}(h^3)$ we will also have from Eq. (5) $\nabla^\nu(T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}) = 0 + \mathcal{O}(h^3)$. So these conservation equations need not be considered separately, because they are embodied in Eqs. (6) and in

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu}. \qquad (7)$$

The content of Eqs. (7) determines the fluid and EM-field perturbations in terms of the propagating GW and will be given below.

The only remaining set of equations that need to be considered is Maxwell's equations $\nabla_\nu F^{\mu\nu} = 4\pi J^\mu$ and

$F_{[\alpha\beta;\gamma]} = 0$. Here the current density is given by $J^\mu = \rho_Q u^\mu + \sigma F^{\mu\nu} u_\nu$. The local charge density $\rho_Q = Z_e n_i - e n_e \simeq 0$, i.e., can be taken as zero due to the mobility of the lighter electron with respect to the other ion species, or due to exact cancelation (i.e., in the case for example of an electron-positron plasma). Also we have $\sigma = 1/(4\pi\eta)$ for the finite conductivity of the plasma fluid, with η the resistivity.

Now from the $(\alpha\beta;\gamma) = (0xy)$ and (xyz)-components of the sourceless Maxwell's equations we exactly obtain Eq. (31) of Appendix A, and

$$\delta B^z(t,z) = \frac{H_0}{A^8}(h_+^2 + h_\times^2), \tag{8}$$

while from the $(0xz)$ and $(0yz)-$components we obtain respectively

$$\begin{aligned}(A^2W^2\delta B^y)_{,t} + A^2(\delta E^x)_{,z} &= 0\\ (A^2W^2\delta B^x)_{,t} - A^2(\delta E^y)_{,z} &= 0\end{aligned} \tag{9}$$

From the sourcefull Maxwell's equations we obtain that $\delta E^z = 0$ and

$$\begin{aligned}&(\delta E^x)_{,t} + (\delta B^y)_{,z} + (\delta E^x)\left[2\frac{\dot{A}}{A} + \frac{\dot{W}}{W} - \right.\\ &\left.-\frac{h_+\dot{h}_+ + h_\times \dot{h}_\times}{A^4} + 2\frac{\dot{A}}{A}\frac{(h_+^2 + h_\times^2)}{A^4}\right] - \\ &-(\delta B^y)\left[\frac{h_+\dot{h}_+ + h_\times\dot{h}_\times}{A^4}\right] + \\ &+4\pi\sigma[\delta E^x + B_0 A^2 \delta u^y] = 0\end{aligned} \tag{10}$$

along with a similar equation, that results from the substitutions $\delta E^x \to \delta E^y$, $\delta B^y \to -\delta B^x$ and $\delta u^y \to -\delta u^x$. These are the propagation equations for the EM-field perturbations.

We now consider Eqs. (7). From the $(0x)$, $(0y)$ and $(0z)$ components we obtain the fluid's four-velocity perturbations

$$\begin{aligned}\delta u^x &= -\frac{1}{4\pi}\frac{A^2 B_0}{(\rho + p)}\delta E^y\\ \delta u^y &= \frac{1}{4\pi}\frac{A^2 B_0}{(\rho + p)}\delta E^x\end{aligned} \tag{11}$$

and

$$\begin{aligned}8\pi(\rho + p)\delta u^z &= \frac{1}{2A^4}(\dot{h}_+ h_+^{'} + \dot{h}_\times h_\times^{'}) + \\ &+ \frac{1}{A^4}(h_+\dot{h}_+^{'} + h_\times \dot{h}_\times^{'}) - \\ &- \left(\frac{\dot{A}}{A} + \frac{\dot{W}}{W}\right)\frac{(h_+ h_+^{'} + h_\times h_\times^{'})}{A^4}\end{aligned} \tag{12}$$

From the (00) component we obtain

$$\begin{aligned}8\pi\delta\rho &= \left(\frac{\dot{A}}{A}\right)^2 \frac{(h_+^2 + h_\times^2)}{A^4} - \\ &- \frac{1}{4}\frac{[(\dot{h}_+)^2 + (\dot{h}_\times)^2]}{A^4} + \\ &+ 2\left(\frac{\dot{A}}{A}\frac{\dot{W}}{W}\right)\frac{(h_+^2 + h_\times^2)}{A^4} + \\ &+ \frac{3}{4W^2}\frac{[(h_+^{'})^2 + (h_\times^{'})^2]}{A^4} - \\ &- \left(\frac{\dot{W}}{W}\right)\frac{(h_+\dot{h}_+ + h_\times\dot{h}_\times)}{A^4} + \\ &+ \frac{1}{W^2A^4}[h_+ h_+^{''} + h_\times h_\times^{''}] - \\ &- 2\frac{H_0 B_0}{A^4}[h_+^2 + h_\times^2]\end{aligned} \tag{13}$$

From proper combinations of the (xx), (yy) and (zz) components of Eqs. (7) we obtain the pressure perturbation

$$\begin{aligned}8\pi\delta p &= -\frac{1}{4W^2}\frac{[(h_+^{'})^2 + (h_\times^{'})^2]}{A^4} + \\ &+ \left[-2\frac{\ddot{A}}{A} + 3\left(\frac{\dot{A}}{A}\right)^2\right]\frac{(h_+^2 + h_\times^2)}{A^4} - \\ &- 4\left(\frac{\dot{A}}{A}\right)\frac{(h_+\dot{h}_+ + h_\times\dot{h}_\times)}{A^4} + \frac{3}{4}\frac{[(\dot{h}_+)^2 + (\dot{h}_\times)^2]}{A^4} + \\ &+ \frac{(h_+\ddot{h}_+ + h_\times\ddot{h}_\times)}{A^4} + 2\frac{H_0B_0}{A^4}[h_+^2 + h_\times^2],\end{aligned} \tag{14}$$

the constraint

$$\begin{aligned}&-\frac{1}{2}\frac{[(\dot{h}_+)^2 + (\dot{h}_\times)^2]}{A^4} + \left(\frac{\dot{A}}{A} + \frac{\dot{W}}{W}\right)\frac{(h_+\dot{h}_+ + h_\times\dot{h}_\times)}{A^4} - \\ &-\frac{1}{W^2A^4}[h_+h_+^{''} + h_\times h_\times^{''}] - \frac{4H_0B_0}{A^4}[h_+^2 + h_\times^2] = 0\end{aligned} \tag{15}$$

and the propagation equation for the *cross* polarization of the gravitational wave

$$\begin{aligned}\Box h_+ &:= \frac{1}{2}\left(\ddot{h}_+ - \frac{1}{W^2}h_+^{''}\right) + \\ &+ 2\left(\frac{\dot{A}}{A}\right)^2 h_+ - \left(\frac{\dot{A}}{A}\right)\dot{h}_+ + \frac{1}{2}\left(\frac{\dot{W}}{W}\right)\dot{h}_+ - \\ &- \left(\frac{\ddot{W}}{W} + 2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}}{A}\frac{\dot{W}}{W}\right)h_+ = \\ &= [8\pi p + A^4(B_0)^2]h_+\end{aligned} \tag{16}$$

Finally from the (xy) component we obtain the propagation equation for the *times* polarization of the GW

$\Box h_\times = [8\pi p + A^4(B_0)^2]h_\times$. Eqs. (8)-(16) constitute a closed system of equations that we study below.

III. ANALYTICAL SOLUTIONS

Substituting the solution of Appendix A into Eq. (16) we obtain (setting also $h_+(t,z) = e^{ik_g z}h_+(t)$)

$$\ddot{h}_+ + \frac{k_g^2}{t^{2l}}h_+ - \left(\frac{2\gamma}{1+\gamma}\right)\frac{\dot{h}_+}{t} + \left(\frac{2\gamma}{1+\gamma}\right)\frac{h_+}{t^2} = 0 \quad (17)$$

A similar equation is satisfied by the other polarization. Also the constraint equation, Eq. (15) is satisfied if both polarizations obey

$$(\dot{h}_+)^2 - \frac{(3-\gamma)}{(1+\gamma)}\frac{h_+\dot{h}_+}{t} - \frac{k_g^2}{t^{2l}}h_+^2 + \frac{(1-\gamma)(3\gamma-1)}{(1+\gamma)^2 t^2}h_+^2 = 0 \quad (18)$$

We can take as final time of our analysis any time before the *recombination* epoch $t_{fin} \simeq 10^{13} sec$, where presumably the cosmological plasma ceases to exist. Also the initial time can be any time after the typical time that inflation starts ($t_0 \simeq 10^{-34} sec$), or the end of the reheating period ($t_0 \simeq 10^{-31} sec$). Between these we recognize two eras: The first is the *small*-time era where in Eq. (16) the *second* term is small compared with the next two. The second is the *large*-time era where the *last* term can be omitted. Equating these we have an estimate of this transition time as

$$t_* = \left[\frac{\gamma}{(1+\gamma)k_g^2}\right]^{1/2(1-l)} \quad (19)$$

We observe that this time scale coincides with the time that the mode, with *comoving* wavevector $k_g = 1/\lambda_c$, enters the Horizon. Indeed the *physical* wavelength $\lambda_{phys} = W\lambda_c$ must be smaller than the Horizon scale, $\lambda_{phys} \le l_H = c/H_W = cW/\dot{W}$. This gives the estimate $t_*^{(H)} = [l^2/k_g^2]^{1/2(1-l)}$, which is of the same order with Eq. (19).

We identify the modes by their wavenumber k_g and their initial-time frequency f_0 so that $2\pi f_0 = k_g$, (we take throughout $c = 1$). At the end of the time interval t_{fin} let the frequency be $f_{fin} = f_m(KHz)$. Due to the cosmological redshift we have $f_0 = f_{fin}W(t_{fin})/W(t_0)$. If we denote $u := (t/t_0)$ then the transition time t_* of Eq. (19) is given by

$$\begin{aligned} log_{10}(u_*) &= \frac{1}{2(1-l)}[(62-162l)+ \\ &+ log_{10}\left(\frac{\gamma}{4\pi^2(1+\gamma)f_m^2}\right)\Big] \end{aligned} \quad (20)$$

and is plotted in Fig. (1). It is evident that this can be obtained for other boundary values for t_0, t_{fin}.

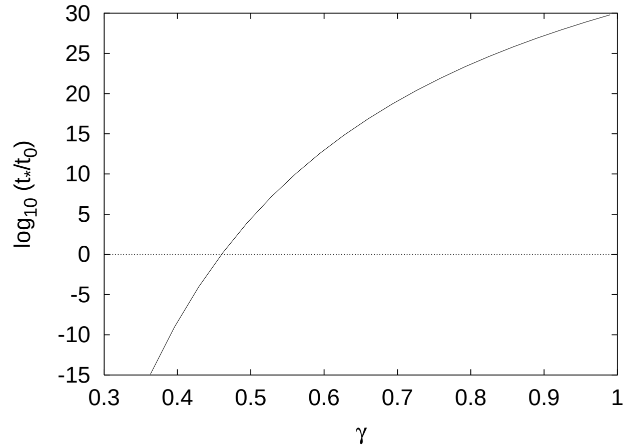

FIG. 1: The transition time t_*, of Eq. (19), as a function of the parameter γ

A. Small-t Solutions

There are many types of solutions, for example power-law solutions, namely when the *second* term on the l.h.s of Eq. (17) is omitted. The first is

$$h_+^{(1)}(t) = C_1 t + C_2 t^{2\gamma/(1+\gamma)} \quad (21)$$

and it is evident that by substituting into Eq. (17) the error is of order $\mathcal{O}(1/t^{2l})$. We also find that Eq. (18) is satisfied with remaining error terms of order $\mathcal{O}(1/t^{2l})$, provided that $C_1 = 0$.

The second solution we found is

$$h_+^{(2)}(t) = C_3 t^{1/2} exp\left[ik_g \int_{t_0}^{t}\frac{du}{u^l}\right] \quad (22)$$

which, when substituted into Eq. (17) gives error terms of order $\mathcal{O}(1/t^2)$! Also the constraint equation Eq. (18) is satisfied to the order $\mathcal{O}(1/t^{l+1})$. Here the mode is outside the Horizon and the amplitude of the GW is assumed to be driven by the so-called cosmological pump field. However, for our purposes to show that certain classes of gravitational waves can induce strong electric field perturbations, we use the following solution

$$h_+(t) = t^{2\gamma/(1+\gamma)} exp\left[bt^l\right] \quad (23)$$

with b a constant. Upon substituting into Eq. (17) we find errors of order $\mathcal{O}(1/t^{2(1-l)})$ and of the same order for Eq. (18).

In order to show the consistency of the whole model we consider Eq. (17) without the first term. This can be solved analytically to obtain the *exact* solution

$$\begin{aligned} h_+(t) &= h_0 texp\left[\frac{k_g^2(1+\gamma)^2}{8\gamma^2}t^{\frac{4\gamma}{(1+\gamma)}}\right] = \\ &= h_0 texp\left[\frac{(1+\gamma)}{8\gamma}(t/t_*)^{4\gamma/(1+\gamma)}\right] \end{aligned} \quad (24)$$

Computing the second derivative we obtain

$$\ddot{h}_+ = \dot{h}_+ \left[\frac{1}{t} + \frac{k_g^2(1+\gamma)}{2\gamma} t^{(3\gamma-1)/(1+\gamma)} \right] + \\ + h_+ \left[-\frac{1}{t^2} + \frac{k_g^2(3\gamma-1)}{2\gamma} \frac{1}{t^{2l}} \right] \quad (25)$$

Thus the seccond derivative will be small with respect to the first and zeroth order derivatives, as times passes, if the second term, in the first set of brackets, is smaller than the first term. This *exactly* reproduces the condition of Eq. (19).

B. Large-t Solution

Here we assume that the last term on the l.h.s of Eq. (17) is omitted and we have the usual solution of a mode that is inside the cosmological Horizon and its amplitude decreases with time,

$$h_+(t) = \frac{1}{t^{2l}} \left(D_1 e^{-ik_g t} + D_2 e^{ik_g t} \right) \quad (26)$$

When substituted into Eq. (17) we have error terms of the order $\mathcal{O}(1/t^{2l+1})$. Also we find that Eq. (18) is satisfied to the order $\mathcal{O}(1/t^{4l})$.

We can match the two solutions of Eqs. (24) and (26) at $t = t_*$ with continuity of the first derivatives and we obtain

$$e^{ik_g t_*} D_2 = \frac{(3/2 + 2l + ik_g t_*)}{2ik_g t_*} h_0(t_*)^{2l+1} exp\left(\frac{1+\gamma}{8\gamma}\right) \\ e^{-ik_g t_*} D_1 = \frac{(-3/2 - 2l + ik_g t_*)}{2ik_g t_*} h_0(t_*)^{2l+1} exp\left(\frac{1+\gamma}{8\gamma}\right) \quad (27)$$

This solution when substituted into Eq. (13) gives the usual oscillating behaviour of the density perturbations when the driver (GW) enters the Horizon, after the time $t = t_*$.

IV. NUMERICAL RESULTS

We consider Eq. (10) and the first of Eqs. (9), where we take the magnetic field perturbations equal to zero for simplicity. We obtain

$$(\delta E^x)_{,u} + (\delta E^x) \left[\frac{2}{(1+\gamma)u} - \frac{2h_+(h_+)_{,u}}{u^2} + \frac{2h_+^2}{u^3} + \\ + 4\pi\sigma_0 \left(1 + \frac{(1-\gamma)(3\gamma-1)}{(1+\gamma)(3-\gamma)} \right) \right] = 0 \quad (28)$$

where the conductivity is in units of $(1/t_0)$. It is evident that the conductivity counteracts the electric field perturbations that arise from the passage of the GW,

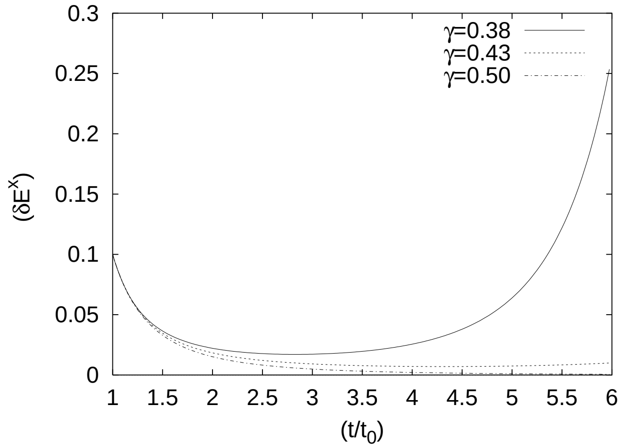

FIG. 2: Electric field perturbation as a function of γ

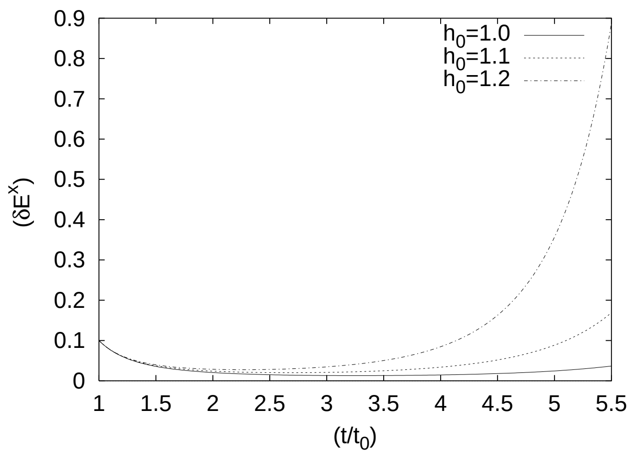

FIG. 3: Electric field perturbation as a function of h_0

whereas the constant b in Eq. (23) acts in favor of steeper in time electric field perturbations. For an initial value of $(\delta E^x)(t_0) = 0.1$ we integrate numerically Eq. (28) for $\sigma_0 = 0.1$, $h_0 = 1.0$ and $b = 1$ and study the dependence of the electic field perturbation on the parameter γ. This is shown in Fig. (2). Also for $\sigma_0 = 0.1$, $b = 1$ and $\gamma = 0.4$ we study the dependence of the electic field perturbation on the parameter h_0, effectively the (normalized) amplitude of the metric perturbations. This behaviour is shown in Fig. (3).

V. DISCUSSION

We have considered the propagation of a gravitational wave perturbation parallel to the ambient magnetic field of an anisotropic magnetized cosmological spacetime, non-linearly coupled with its temporal derivatives. It is known that in the first order (linear) approximation with respect to the amplitude of the perturbation, no electric field or density perturbations arise. However in the

non-linear level this behaviour changes. Namely while certain classes of power-law solutions, for these metric perturbations, *cannot* induce increasing in time electric field perturbations, certain other classes of solutions can do so.

Our numerical results and their theoretical counterparts show that while power-law solutions, such as those of Eqs. (21) and (22), are not capable of inducing increasing in time, electric field perturbations, solutions of the form of Eqs. (23) and (24) can do so. Moreover these solutions are consistent with the *small-t* and *large-t* assumptions, as these are encoded in Eq. (19). The consistency of the whole theoretical model is further shown by the numerical treatment of Eq. (10). The electric field perturbations depend on four parameters: First on the conductivity σ_0, which in an evident and expected manner counteracts the generation of electric field perturbations. Second on the constant b, in the class of solutions of Eq. (23) which again in an expected manner acts in favour of steeper in time electric field and density perturbations. The (normalized) amplitude h_0 of the metric perturbations is the third parameter, as it is shown in Fig. (3) and γ is the fourth parameter, as it is shown in Fig. (2). Since the Hubble parameter in the $z-$direction is $H_z := (\frac{\dot{W}}{W}) = (1-\gamma)/(1+\gamma)t$ we see that higher values of the $\gamma-$parameter imply smaller values of this parameter, which in turn acts in favour of the electric field perturbations.

Our results hopefully shed some light on the full non-linear behaviour of conductive plasmas, in anisotropic cosmological models, coupled with an ambient magnetic field and driven by (not necessarily weak) metric perturbations.

Acknowledgments

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Appendix A

We present here for convenience the class of anisotropic magnetized cosmologies, with perfect fluid content, known as Thorne's models [3]. The scale factors are given by $A(t) = t^{1/2}$ and $W(t) = t^l$ as used in Eq. (1). Here

$$l := \frac{1-\gamma}{1+\gamma}, \quad (\frac{1}{3} < \gamma \leq 1) \tag{29}$$

The matter energy density and pressure are given by

$$\rho(t) = \frac{3-\gamma}{16\pi(1+\gamma)^2 t^2}, \quad p(t) = \gamma\rho(t) \tag{30}$$

Finally the magnetic field points in the $z-$direction and is given by

$$B_0(t) = \frac{H_0}{t^2}, \quad H_0 := \frac{(1-\gamma)^{1/2}(3\gamma-1)^{1/2}}{2(1+\gamma)} \tag{31}$$

Also we note that the relativistic Alfven velocity is $u_A^2 = v_A^2/(1+v_A^2)$ where in conformity with the notation used in [3] we have

$$v_A^2 := \frac{(F_y^x)^2}{4\pi\rho} = \frac{(1-\gamma)(3\gamma-1)}{(3-\gamma)} \tag{32}$$

Appendix B

We present here two of the components of Einstein's tensor, just for the purpose of displaying the relative complexity of the task, computed from Eq. (1), after they have been carefully expanded up to the *second* order with respect to the GW amplitude. This is a difficult task and has been performed with great care. We obtain

$$\begin{aligned} G_{00} &= \left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{\dot{A}\dot{W}}{AW}\right) + \left(\frac{\dot{A}}{A}\right)^2 \left[\frac{h_+^2}{A^4} + \frac{h_\times^2}{A^4}\right] \\ &- \frac{1}{4}\left[\frac{(\dot{h}_+)^2}{A^4} + \frac{(\dot{h}_\times)^2}{A^4}\right] \\ &+ 2\left(\frac{\dot{A}\dot{W}}{AW}\right)\left[\frac{h_+^2}{A^4} + \frac{h_\times^2}{A^4}\right] \\ &+ \frac{3}{4W^2}\left[\frac{(h_+^{'})^2}{A^4} + \frac{(h_\times^{'})^2}{A^4}\right] \\ &- \left(\frac{\dot{W}}{W}\right)\left[\frac{h_+\dot{h}_+}{A^4} + \frac{h_\times\dot{h}_\times}{A^4}\right] \\ &+ \frac{1}{W^2A^4}[h_+h_+^{''} + h_\times h_\times^{''}] \end{aligned} \tag{33}$$

and

$$\begin{aligned} G_{xx} &= A^2\left[-\frac{\ddot{A}}{A} - \frac{\ddot{W}}{W} - \frac{\dot{A}\dot{W}}{AW}\right] + \frac{1}{2}\left(\ddot{h}_+ - \frac{1}{W^2}h_+^{''}\right) + \\ &+ 2\frac{(\dot{A})^2}{A^2}h_+ - \left(\frac{\dot{A}}{A}\right)\dot{h}_+ + \frac{1}{2}\frac{\dot{W}}{W}\dot{h}_+ - \\ &- \left(\frac{\ddot{W}}{W} + 2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}\dot{W}}{AW}\right)h_+ + \\ &+ \frac{1}{4}\left[\frac{(\dot{h}_+)^2 + (\dot{h}_\times)^2}{A^2}\right] - 3\left(\frac{\dot{A}}{A}\right)\frac{h_+\dot{h}_+ + h_\times\dot{h}_\times}{A^2} + \\ &+ \left(\frac{\dot{W}}{W}\right)\frac{h_+\dot{h}_+ + h_\times\dot{h}_\times}{A^2} - \frac{1}{4W^2}\left[\frac{(h_+^{'})^2 + (h_\times^{'})^2}{A^2}\right] + \\ &+ \left[-2\left(\frac{\ddot{A}}{A}\right) - 2\frac{\dot{A}\dot{W}}{AW} + 3\left(\frac{\dot{A}}{A}\right)^2\right]\frac{h_+^2 + h_\times^2}{A^2} \\ &+ \frac{h_+}{A^2}\left[\ddot{h}_+ - \frac{h_+^{''}}{W^2}\right] + \frac{h_\times}{A^2}\left[\ddot{h}_\times - \frac{h_\times^{''}}{W^2}\right] \end{aligned} \tag{34}$$

Here a dot denotes derivative with respect to the time t, while a prime denotes derivative with respect to the spatial variable z. Similar expressions are obtained for the rest of the non-zero components, namely for G_{yy}, G_{zz}, G_{xy} and G_{0z}.

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